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Towards an Inclusive, Safe, Resilient and Sustainable City: Approaches and Tools

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INVESTIGATING CONDITIONS ENSURING RELIABILITY OF THE PRIORITY VECTORS

Bice Cavallo, Livia D'Apuzzo, Luciano Basile

Abstract

In this paper, we investigate conditions, weaker than consistency, that a pairwise comparison matrix has to satisfy in order to ensure that priority vectors proposed in literature are ordinal evaluation vectors for the actual ranking.
In particular, we introduce a partial order on the rows of a pairwise comparison matrix; if it is a simple order, then the matrix is transitive, the actual ranking is easily established and priority vectors are ordinal evaluation vectors for the actual ranking.

Keywords: pairwise comparison matrices, ordinal evaluation vectors, simple order

INDAGARE LE CONDIZIONI CHE ASSICURINO L'AFFIDABILITÀ DEI VETTORI PRIORITÀ

Sommario

In questo articolo, analizziamo le condizioni, più deboli della consistenza, che una matrice di confronti a coppie dovrebbe soddisfare affinché i vettori priorità proposti in letteratura siano vettori di valutazione ordinale.
In particolare, introduciamo una relazione di ordine parziale sulle righe di una matrice di confronti a coppie; se tale relazione rappresenta un ordine semplice, allora la matrice è transitiva, ed è possibile stabilire in maniera semplice l'effettivo ordinamento e i vettori priorità sono vettori di valutazione ordinale.

Parole chiave: matrici di confronto a coppie, vettori di valutazione ordinale, relazione di ordine semplice

1. Introduction

Most decision processes related to planning, territory government, technology transfer, transportation, conflict resolution etc. involve a multiplicity of criteria and sub-objectives (e.g. economic and social), the satisfaction of which is crucial in building the best alternative.
The pairwise comparisons are an essential tool to establish the relative importance of criteria or sub-objectives that are measurable in different scales. In fact, they constitute the crucial tool of the Analytic Hierarchy Process (AHP) (Saaty, 1977, 1980, 1986, 2008), a Multi-Criteria method introduced by Saaty (1977) for evaluating alternatives.
The AHP organizes the elements of the decision process in a hierarchy and uses the pairwise comparisons for getting a weighted ranking of the elements of a level with respect to an element in the upper level; then the local weights of the elements of each level are combined to get the global weights of the alternatives.
Unfortunately, it may happen that the methods proposed in literature for obtaining weighted rankings for alternatives/criteria are not reliable. In this paper, we focus on this problem and propose a condition that ensures the reliability of these methods.
The paper is organized as follows: in Section 2, we introduce Multiplicative Pairwise Comparison Matrices (MPCMs) and a partial order $\trianglerighteq$ on the rows of a matrix, we focus on transitive matrices and consistent matrices and show that if a matrix is transitive, but not consistent, then it may be that priority vectors proposed in literature are not reliable; in Section 3, we prove that if $\trianglerighteq$ is a simple order, then the matrix is transitive and the priority vectors provide reliable weighted ranking; in Section 4, we provide concluding remarks and directions for future work.

2. Multiplicative pairwise comparison matrices and priority vectors

Let $X = \{x_1, x_2, ..., x_n\}$ be a set of decision elements such as criteria or alternatives and

$$A = (a_{ij}) = \begin{pmatrix} 1 & a_{12} & ... & a_{1n} \\ a_{21} & 1 & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & 1 \end{pmatrix} \tag{1}$$

the related MPCM. Thus, the entry a_{ij} is a positive number that represents the preference ratio of x_i over x_j : so $a_{ij} = 1$ if and only if there is indifference between x_i and x_j , $a_{ij} > 1$ if and only if x_i is strictly preferred to x_j , whereas $a_{ij} < 1$ expresses the reverse preference.
For an algebraic approach to pairwise comparison matrices, see Cavallo (2014), Cavallo and D'Apuzzo (2009, 2010, 2012, 2014) and Cavallo *et al.* (2012).
For MPCMs, the following condition of reciprocity:

$$a_{ji} = \frac{1}{a_{ij}} \quad \forall i, j \in \{1,2,\ldots,n\} \tag{2}$$

is assumed.
Under assumption of reciprocity, we set:

$$x_i \succ x_j \Leftrightarrow a_{ij} > 1, \qquad x_i \sim x_j \Leftrightarrow a_{ij} = 1\,, \tag{3}$$

where $x_i \succ x_j$ and $x_i \sim x_j$ stand for "x_i is strictly preferred to x_j" and "x_i and x_j are indifferent", respectively.
Moreover, we set:

$$x_i \succsim x_j \Leftrightarrow (x_i \succ x_j \ or \ x_i \sim x_j) \Leftrightarrow a_{ij} \geq 1, \tag{4}$$

that stands for "x_i is weakly preferred to x_j".
The relation $\succ$ is asymmetric, the relation $\sim$ is reflexive and symmetric and

$$x_i \succ x_j \quad or \quad x_i \sim x_j \quad or \quad x_j \succ x_i \quad \forall i,j \in \{1,2,\ldots,n\}. \tag{5}$$

The relation $\succsim$ is strongly complete, that is:

$$x_i \succsim x_j \quad or \quad x_j \succsim x_i \quad \forall i,j \in \{1,2,\ldots,n\}; \tag{6}$$

thus, if $\succsim$ is a transitive relation, then $\succsim$ is a weak order (Roberts, 1979).
The transitivity of $\succsim$ is the minimal logical requirement and a fundamental principle that preference relations should satisfy; the transitivity is in fact acyclic about the alternatives or criteria ranking. If $\succsim$ is transitive, then there is a rearrangement $(i_1, i_2, \cdots, i_n)$ of $\{1,2,\cdots,n\}$ such that:

$$x_{i_1} \succsim x_{i_2} \succsim \ldots \succsim x_{i_n}. \tag{7}$$

We call (7) the actual ranking on X.

Order relations on the rows set of $A = (a_{ij})$

Let $\underline{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ be the i-th row of $A = (a_{ij})$ and $R_A = \{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$ the rows set of $A = (a_{ij})$. Then, we consider the following order relations:

– $\triangleright$ the strict partial order (i.e. $\triangleright$ is transitive and asymmetric; see Roberts, 1979) on R_A defined by:

$$\underline{a}_r \triangleright \underline{a}_s \Leftrightarrow a_{rj} > b_{sj}, \forall j \in \{1,2,\ldots,n\}; \tag{8}$$

– $\trianglerighteq$ the partial order (i.e. $\trianglerighteq$ is reflexive, antisymmetric and transitive, see (Roberts, 1979)) on R_A defined by:

$$\underline{a}_r \trianglerighteq \underline{a}_s \Leftrightarrow \underline{a}_r \triangleright \underline{a}_s \ or \ \underline{a}_r = \underline{a}_s. \tag{9}$$

We stress that if $\trianglerighteq$ is strongly complete, that is:

$$\forall \underline{a}_r, \underline{a}_s \in R_A \quad \underline{a}_r \trianglerighteq \underline{a}_s \ or \ \underline{a}_s \trianglerighteq \underline{a}_r, \tag{10}$$

then $\trianglerighteq$ is a simple order see (Roberts, 1979).

Transitive MPCMs and ordinal evaluation vectors
Cavallo and D'Apuzzo (2014) provide the notion of transitivity for a matrix defined over an abelian linearly ordered group; by considering MPCMs, we have the following definition:

Definition 1
$A = (a_{ij})$ transitive if and only

$$a_{ij} \geq 1 \quad a_{jk} \geq 1 \Rightarrow a_{ik} \geq 1. \tag{11}$$

Let $A=(a_{ij})$ be a reciprocal MPCM. By reciprocity, implication in (11) is equivalent to the following implications:

$$a_{ij}>1 \quad a_{jk}>1 \Rightarrow a_{ik}>1, \qquad a_{ij}=1 \quad a_{jk}=1 \Rightarrow a_{ik}=1. \tag{12}$$

Proposition 1
$A=(a_{ij})$ is transitive if and only if $\succsim$ is a transitive relation.
Proof.
By Definition 1 and equation(4).

Thus, if $A=(a_{ij})$ is transitive, the actual ranking on X is achievable.

Definition 2 (Cavallo and D'Apuzzo, 2014)
Let $A=(a_{ij})$ be transitive. A positive vector $\underline{w}=(w_1,w_2,\ldots,w_n)$ is an ordinal evaluation vector for the ranking in (7) if and only if

$$w_i > w_j \Leftrightarrow x_i \succ x_j \quad \textit{and} \quad w_i = w_j \Leftrightarrow x_i \sim x_j,$$

or, equivalently:

$$w_i \geq w_j \Leftrightarrow x_i \succsim x_j.$$

Consistent MPCMs
In an ideal situation, in which the Decision Maker is strongly coherent when stating his/her preferences, Cavallo and D'Apuzzo (2014) provide the notion of consistency for a matrix defined over an abelian linearly ordered group; by considering MPCMs, we have the following condition:

$$a_{ij}a_{jk}=a_{ik} \quad \forall i,j,k \in \{1,2,\ldots,n\}. \tag{13}$$

Under assumption of reciprocity in (2), the consistency condition in (13) implies the transitivity condition in (11) (Cavallo and D'Apuzzo, 2014) and, as a consequence, the actual ranking is established; the reverse implication does not hold (e.g. the MPCM in Example 1 is transitive but no consistent).
Brunelli and Fedrizzi (2014) analyze some inconsistency indices for MCPMs, and Chiclana *et al.* (2009) analyze consistency of fuzzy pairwise comparison matrices.

Example 1
Let us consider the set $X=\{x_1,x_2,x_3,x_4\}$ and the related MPCM:

$$A=\begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{1}{2} & 1 & 1 & 2 \\ \frac{1}{3} & 1 & 1 & 5 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{5} & 1 \end{pmatrix}.$$

By inequality $a_{1j} > 1$, for each $j \in \{2,3,4\}$, x_1 is strictly preferred to each other x_j; by equality $a_{23} = 1$, x_2 and x_3 are indifferent; by inequalities $a_{24} > 1$ and $a_{34} > 1$, x_2 and x_3 are strictly preferred to x_4. Thus, the relation $\succsim$ is transitive (i.e. $A = (a_{ij})$ is transitive) and the actual ranking on X is $x_1 \succ x_2 \sim x_3 \succ x_4$.
However, $A = (a_{ij})$ is no consistent (e.g. $a_{13}a_{34} \neq a_{14}$).

The following proposition shows that the consistency condition is equivalent to the proportionality of the rows, and implies that $\trianglerighteq$ is a simple order on the rows.

Proposition 2
The following assertions hold:
1. $A = (a_{ij})$ is consistent if and only if

$$\underline{a}_i = a_{ij}\underline{a}_j \quad \forall i, j \in \{1,2,\ldots,n\}; \tag{14}$$

2. if $A = (a_{ij})$ is consistent, then $\trianglerighteq$ is strongly complete.

Proof.
Equation (13) is equivalent to:

$$\frac{a_{ik}}{a_{jk}} = a_{ij}, \quad \forall i, j, k \in \{1,2,\ldots,n\},$$

that is equivalent to (14).
By (14) and $a_{ij} > 0$, we have:

$$a_{ij} > 1 \Leftrightarrow \underline{a}_i \triangleright \underline{a}_j \quad a_{ij} = 1 \Leftrightarrow \underline{a}_i = \underline{a}_j \quad a_{ij} < 1 \Leftrightarrow \underline{a}_j \triangleright \underline{a}_i,$$

thus, (10) holds.

Priority vectors
In literature, several methods have been proposed to build priority vectors, that are positive vectors $\underline{w} = (w_1, w_2, \ldots, w_n)$ assigning a preference order on X by means of the relations $\succ_{\underline{w}}$ and $\sim_{\underline{w}}$ defined by the following equivalences:

$$x_i \succ_{\underline{w}} x_j \Leftrightarrow w_i > w_j \quad \textit{and} \quad x_i \sim_{\underline{w}} x_j \Leftrightarrow w_i = w_j. \tag{15}$$

Then, given a priority vector $\underline{w} = (w_1, w_2, \ldots, w_n)$, a weighting vector (providing the weights for the decision elements $x_1, x_2, \ldots, x_n$) is the following one:

$$\underline{w}^* = \frac{1}{\sum_{i=1}^{n} w_i} \underline{w}$$

obtained by normalizing $\underline{w}$ up to 1. The vector $\underline{w}^*$ is also called priority dominance vector.

Of course, whenever $A=(a_{ij})$ is transitive, a priority vector is reliable if and only if $\succ_{\underline{w}}$ and $\sim_{\underline{w}}$ coincide with $\succ$ and $\sim$, respectively.

The most used methods for deriving priority vectors from a MPCM are the eigenvector method and the geometric or arithmetic mean (Saaty, 1977; 1980; 1986; 2008; Barzilai, 1997) that provide:

- a right positive eigenvector $\underline{w}_{\lambda_{max}}$ associated with the greatest eigenvalue λ_{max} of $A=(a_{ij})$, that is a positive vector solution of equation $A\underline{w}=\lambda_{max}\underline{w}$;
- the arithmetic mean vector $\underline{w}_{am}=(\frac{1}{n}\sum_{j=1}^{n}a_{1j},\frac{1}{n}\sum_{j=1}^{n}a_{2j},\ldots,\frac{1}{n}\sum_{j=1}^{n}a_{nj})$;
- the geometric mean vector $\underline{w}_{gm}=(\prod_{j=1}^{n}a_{1j}^{\frac{1}{n}},\prod_{j=1}^{n}a_{2j}^{\frac{1}{n}},\ldots,\prod_{j=1}^{n}a_{nj}^{\frac{1}{n}})$.

Under consistency condition in (13), $\underline{w}_{\lambda_{max}}$, $\underline{w}_{am}$ and $\underline{w}_{gm}$ are reliable vectors, because provide a preference order on X equal to the actual ranking.

Unfortunately, condition (13) is hard to reach in real situations; thus, it may happen that $\underline{w}_{\lambda_{max}}$, $\underline{w}_{am}$ and $\underline{w}_{gm}$ are not reliable because they provide a preference order on X different from the actual ranking (see Example 2).

Example 2

Let us consider the MPCM in Example 1. The vectors $\underline{w}_{\lambda_{max}}=(0.82,0.36,0.43,0.15)$, with $\lambda_{max}=4.177$, $\underline{w}_{am}=(2.5,1.12,1.8,0.49)$ and $\underline{w}_{gm}=(2.13,1,1.14,0.4)$ provide the ranking $x_1\succ_{\underline{w}}x_3\succ_{\underline{w}}x_2\succ_{\underline{w}}x_4$ that does not coincide with the actual ranking; so they are not ordinal evaluation vectors.

3. Property of $\trianglerighteq$ ensuring reliability of priority vectors

At the light of the previous considerations, this section aims at establishing a condition stronger than transitivity, but weaker than consistency, under which $\underline{w}_{\lambda_{max}}$, $\underline{w}_{am}$ and $\underline{w}_{gm}$ are ordinal evaluation vectors.

Proposition 3

Let $\trianglerighteq$ be strongly complete. Then, the following equivalences hold:

$$a_{ij}>1\Leftrightarrow\underline{a}_i\triangleright\underline{a}_j\qquad a_{ij}=1\Leftrightarrow\underline{a}_i=\underline{a}_j.$$

Proof.

Let $a_{ij}>1=a_{jj}$. Then, $\underline{a}_i\neq\underline{a}_j$ and, as $\trianglerighteq$ is strongly complete, we get $\underline{a}_i\triangleright\underline{a}_j$. Viceversa, if $\underline{a}_i\triangleright\underline{a}_j$ then $a_{ik}>a_{jk}$ for each k, in particular, for $k=j$, we have $a_{ij}>a_{jj}=1$.

Let $a_{ij}=1=a_{jj}$. Then, as $\trianglerighteq$ is strongly complete, we get $\underline{a}_i=\underline{a}_j$. Viceversa, if $\underline{a}_i=\underline{a}_j$ then $a_{ik}=a_{jk}$ for each k, in particular, for $k=j$, we have $a_{ij}=a_{jj}=1$.

Theorem 1

Let $\trianglerighteq$ be strongly complete. Then, $A=(a_{ij})$ is transitive and $\underline{w}_{\lambda_{max}}$, $\underline{w}_{am}$ and $\underline{w}_{gm}$ are ordinal evaluation vectors for the actual ranking.

Proof.

Let $a_{ij}>1$ and $a_{jk}>1$. By Proposition 3 and transitivity of $\triangleright$, we get $\underline{a}_i \triangleright \underline{a}_j \triangleright \underline{a}_k$. Thus, $a_{ir}>a_{kr}$, for each $r\in\{1,\dots,n\}$; in particular, for $r=k$, $a_{ik}>a_{kk}=1$.

Let $a_{ij}=1$ and $a_{jk}=1$. By Proposition 3, we get $\underline{a}_i=\underline{a}_j=\underline{a}_k$. Thus, $a_{ir}=a_{kr}$, for each $r\in\{1,\dots,n\}$; in particular, for $r=k$, $a_{ik}=a_{kk}=1$.

Thus, by (12), $A=(a_{ij})$ is transitive.

Let us denote by w_i, with $i\in\{1,\dots,n\}$, the i-th component of the vector $\underline{w}_{\lambda_{max}}$, then, by $A\underline{w}_{\lambda_{max}}=\lambda_{max}\underline{w}_{\lambda_{max}}$, we have:

$$w_i=\frac{1}{\lambda_{max}}\sum_{k=1}^{n}a_{ik}w_k. \tag{16}$$

Let us denote by v_i and u_i, with $i\in\{1,\dots,n\}$, the i-th component of the vectors $\underline{w}_{am}$ and $\underline{w}_{gm}$, respectively.

Let $a_{ij}>1$. By Proposition 3, $\underline{a}_i \triangleright \underline{a}_j$, and as a consequence, we have:

$$\sum_{k=1}^{n}a_{ik}>\sum_{k=1}^{n}a_{jk},\quad \prod_{k=1}^{n}a_{ik}>\prod_{k=1}^{n}a_{jk};$$

thus, $v_i>v_j$ and $u_i>u_j$. Moreover, as $w_i>0$, we have:

$$\sum_{k=1}^{n}a_{ik}w_k>\sum_{k=1}^{n}a_{jk}w_k;$$

thus, by $\lambda_{max}>0$ and (16), $w_i>w_j$.

Viceversa, let $v_i>v_j$ (resp. $w_i>w_j$ and $u_i>u_j$). If ad absurdum $a_{ij}\le 1$ then, by reciprocity, $a_{ji}\ge 1$. Thus, by Proposition 3, we get $\underline{a}_j \trianglerighteq \underline{a}_i$ and, as a consequence $v_j\ge v_i$ (resp. $w_j\ge w_i$ and $u_j\ge u_i$), against the assumption.

Let $a_{ij}=1$. By Proposition 3, $\underline{a}_i=\underline{a}_j$ and, as a consequence, we have:

$$\sum_{k=1}^{n}a_{ik}=\sum_{k=1}^{n}a_{jk},\quad \prod_{k=1}^{n}a_{ik}=\prod_{k=1}^{n}a_{jk};$$

thus, $v_i=v_j$ and $u_i=u_j$. Moreover, as $w_i>0$, we have:

$$\sum_{k=1}^{n}a_{ik}w_k=\sum_{k=1}^{n}a_{jk}w_k;$$

thus, by $\lambda_{max} > 0$ and (16), $w_i = w_j$.

Viceversa, let $v_i = v_j$ (resp. $w_i = w_j$ and $u_i = u_j$). If ad absurdum $a_{ij} > 1$ or $a_{ij} < 1$, then, by Proposition 3, $\underline{a}_i \rhd \underline{a}_j$ or $\underline{a}_j \rhd \underline{a}_i$ and, as a consequence $v_i > v_j$ or $v_j > v_i$ (resp. ($w_i > w_j$ or $w_j > w_i$) and ($u_i > u_j$ or $u_j > u_i$)), against the assumption.

Thus, by Definition 2, $\underline{w}_{\lambda_{max}}$, $\underline{w}_{am}$ and $\underline{w}_{gm}$ are ordinal evaluation vectors for the actual ranking.

Of course, by Theorem 1 and Proposition 3, if $\trianglerighteq$ is strongly complete, then the following equivalence holds:

$$(\underline{a}_{i_1} \trianglerighteq \underline{a}_{i_2} \trianglerighteq \ldots \trianglerighteq \underline{a}_{i_n}) \Leftrightarrow (x_{i_1} \succsim x_{i_2} \succsim \ldots \succsim x_{i_n}). \tag{17}$$

Example 3

Let us consider the MPCM

$$A = \begin{pmatrix} 1 & 1 & 3 & 5 \\ 1 & 1 & 3 & 5 \\ \frac{1}{3} & \frac{1}{3} & 1 & 4 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{4} & 1 \end{pmatrix}.$$

R_A is totally ordered by $\trianglerighteq$; indeed: $\underline{a}_1 = \underline{a}_2 \rhd \underline{a}_3 \rhd \underline{a}_4$. Thus, the actual ranking is

$$x_1 \sim x_2 \succ x_3 \succ x_4.$$

Let us stress that $A = (a_{ij})$ is no consistent because (14) is not verified (e.g. the rows $\underline{a}_2$ and $\underline{a}_3$ are not proportional among them).

Finally, the vectors $\underline{w}_{\lambda_{max}} = (0.67, 0.67, 0.28, 0.11)$, with $\lambda_{max} = 4.097$, $\underline{w}_{am} = (2.5, 2.5, 1.42, 0.41)$ and $\underline{w}_{gm} = (1.97, 1.97, 0.82, 0.32)$ are ordinal evaluation vectors.

4. Conclusions and future work

We introduce a partial order $\trianglerighteq$ on the rows set of a Multiplicative Pairwise Comparison Matrix $A = (a_{ij})$; if $\trianglerighteq$ is a simple order, then $A = (a_{ij})$ is transitive and the right positive eigenvector $\underline{w}_{\lambda_{max}}$, the arithmetic mean vector $\underline{w}_{am}$ and the geometric mean vector $\underline{w}_{gm}$ are ordinal evaluation vectors for the actual ranking.

The ranking on the rows, obtained by means of $\trianglerighteq$, allows us to state the actual ranking on the set X of alternatives/criteria. Moreover, the condition of being $\trianglerighteq$ a simple order is weaker than consistency.

Our future work will be directed to investigate the existence of conditions weaker than simple order ensuring that at least one vector among $\underline{w}_{\lambda_{max}}$, $\underline{w}_{am}$ and $\underline{w}_{gm}$ is still an ordinal evaluation vector.

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