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Sezione *Evolving Philosophy*

Music: numbers in motion[*]

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Music is a hidden arithmetic exercise of the soul, which does not know that it is counting.
Gottfried Wilhelm von Leibniz[***]

1. Introduction

One often hears comments to the effect that mathematics and music have much in common. This is clearly (and to some extent uninterestingly) true since music (and rhythm in particular) embodies the notion of counting. So, at least as far as counting goes, there is an obvious connection between mathematics and music. It is also obvious that there is some additional connection that derives from the fact that pitches can be described numerically, and so we see that there are at least two ways in which mathematics and music intersect.

Indeed, the connection between mathematics and music is a deeper one that takes us to the origin of Western thought. Even though it seems quite possible that the connection probably predates even the Greeks, it is with the Greeks that we will begin.

According to medieval scholars, the liberal arts were divided into two subsets: the *quadrivium*, and the *trivium* (these are Latin words that indicate crossroads made of, respectively, four and three roads). The *trivium* was considered the lower division of the liberal arts and consisted of grammar, rhetoric, and logic, namely those skills that were necessary for successful argumentation, and was considered preliminary to theology and philosophy. While the term *quadrivium* was used in this sense for the first time by Boethius, in *De institutione aritmethica*, and was incorporated in medieval thinking, it truly had its origin in Greek philosophy, dating back to Plato's *Republic* if not earlier. It consists of the organization of learning according to four mathematical disciplines. Indeed, in his *Commentary on the First Book of Euclid's Elements*, Proclus comments on the Pythagorean's approach to learning in a way that offers the most precise definition of the *quadrivium*:

> The Pythagoreans considered all mathematical science to be divided into four parts: one half they marked off as concerned with quantity, the other half with magnitude; and each of these they posited as two-fold. A quantity can be considered in regard to its character by itself or in its relation to another quantity, magnitudes as either stationary or in motion. Arithmetic, then, studies quantities as such, music the relations between quantities, geometry magnitude at rest, spherics [astronomy] magnitude inherently moving.[1]

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[***] Letter to Christian Goldbach, April 17, 1712.
[1] Proclus, *A commentary on the first book of Euclid's Elements*, xii, trans. G.R. Morrow, NJ, Princeton University Press, Princeton, 1992, pp. 29-30.

This description of mathematics is sometimes summarized by the following matrix, where the rows are numbers and shapes (alternative ways of expressing quantity and magnitude) and the columns are at rest and in motion.

	At rest	In Motion
Numbers	Arithmetic	Music
Shapes	Geometry	Astronomy

Figure 1.

So, here we have it. For the Greeks, mathematics was composed of four subdisciplines, and music was one of those! It is worth pointing out that the word mathematics is, in itself, a loaded word, as it takes its etymology from the Greek word *μανθάνειν*, which means to learn. So, the *quadrivium* is nothing but mathematics, which in turn is nothing but learning par excellence. And as a portion of that learning we find music, in close connection with numbers (numbers in motion) and with astronomy (both are disciplines of motion ... an observation that is relevant in the context of the ancient and medieval notion of music of the spheres). And of course the fact that music was considered to be the study of numbers in motion is a direct interpretation of our early comments on rhythm and on the fact that every note (pitch) can be represented by a number.

This Pythagorean-Platonic conception of music will reach the Renaissance through Boethius' treaty *De institutione musica*, in which its author follows the Platonic theories according to which the soul of the world conforms to musical harmony. The medieval culture, steeped in the astronomical-musical conception of the music of the spheres, sees arithmetics, the science of numbers, as the foundation for music.[2]

[2] See L. Wuidar, "L'interdetto della conoscenza. Segreti celesti e arcani musicali nel Cinquecento e nel Seicento", *Bruniana & Campanelliana* XV, 1 (2009): pp. 135-152; N. Fabbri, *"De l'utilité de l'harmonie". Filosofia, scienza e musica in Mersenne, Descartes e Galileo,* Edizioni della Normale, Pisa, 2008.

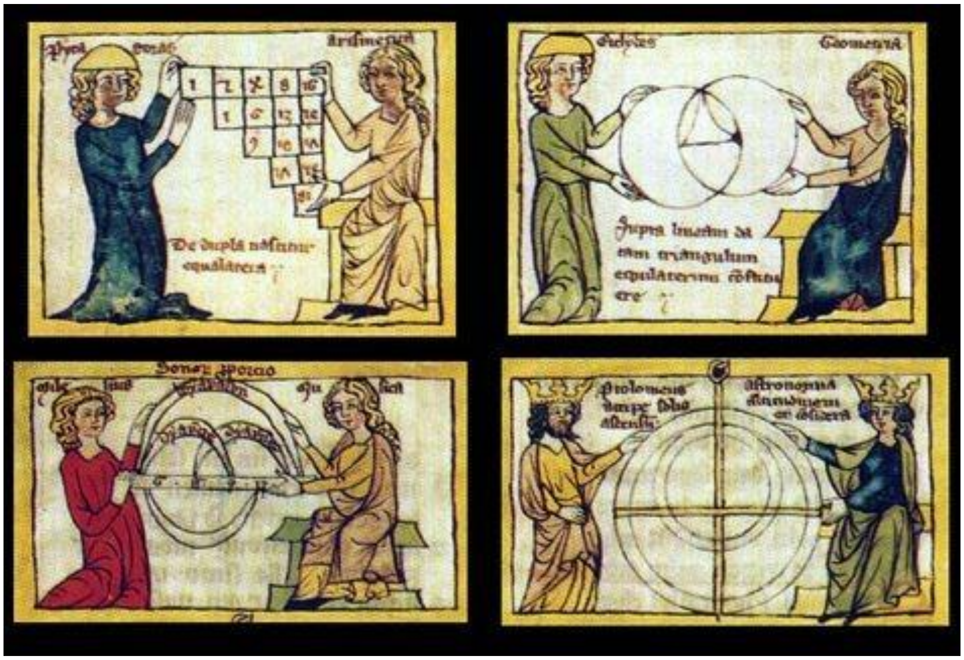

Figure 2.
Top Left: Arithmetic. Top Right: Geometry. Bottom Left: Music. Bottom Right: Astronomy. Excerpt from Thomasin von Zirclaere (Tommasino Di Cerclaria), *Der welhische gast*, Ostfranken (?), 1340. Gotha, Forschungsbibliothek, Cod. Memb. I 120 (Sigle: G), fol. 65v.

But where is that we find a first, early, understanding of the ability to represent pitches with numbers? And why was that a philosophically important issue?

As legend goes, Pythagoras had for a long time tried to understand how consonant sounds can be produced and Boethius gives us a well known (though probably apocryphal) anecdote,[3] according to which Pythagoras was walking in front of a forge, when he heard that the sounds of the hammers were creating different harmonies. At first he believed that it was the different strength of the workers to cause different sounds; to better understand the phenomenon he asked the workers to exchange the hammers, and realized that the property of the sounds did not depend on the muscles of the men, but rather by the weight of the hammers. According to the legend, he realised that hammers that were in a (2 : 1) ratio resonated in perfect consonance (octave). He also realised that other important ratios were (4 : 3), (3 : 2) and (9 : 8). This particular ratios, which musically correspond to the intervals of the octave, the fourth, the fifth and the tone, are the foundations for ancient music and can indeed be empirically constructed

[3] The anecdote was probably initially transmitted by Nicomachus of Gerasa, *The Manual of Harmonics of Nicomachus the Pythagorean*, F.R. Levin (ed.), Phanes Press, Grand Rapids (MI), 1994, ch. 6, p. 83. The narration had a large fortune and was resumed in Iamblichus, *De vita pythagorica liber*, L. Deubner (ed.), Teubner, Leipzig, 1937, ch. xxvi; it then became exemplary in the Boethian pages of the *De institutione musica libri quinque*, 1, 10.

by sectioning a string according to those ratios (rather than by using different hammers, for physical reasons that we do not need to clarify here).[4]

To better understand how the Greeks employed these particular ratios, we will go back to one of the most studied Platonic dialogues, a dialogue that fascinated the scholars of the middle-ages, and that has not stopped fascinating modern scholars as well. We are talking here about the *Timaeus*, a beautiful dialogue to which we devote the next few pages.

2. The canon in Plato's *Timaeus*

The dialogue opens with an enigmatic count, that foregrounds the discussion to come. As in all dialogues, Socrates is the center of the scene, and in this specific case his major interlocutor is Timaeus, who is introduced to us immediately as a Pythagorean from what is now the southern Italian town of Locri (of course, at that time Locri would have been part of the Magna Grecia, not too far away from where Pythagoras had set up shop in Crotone, both towns being on the Ionian coast of Calabria, the southernest region of continental Italy).

So, let us read the opening of the dialogue:

> *Socrates*: One, two, three; but where, my dear Timaeus, is the fourth of our guests of yesterday, our hosts of to-day?
> *Timaeus*: Some sickness has befallen him, Socrates; for he would never have stayed away from our gathering of his own free will.[5]

While, apparently, these words refer to the number of participants in the dialogue, and seem to have a simple introductory significance, we will soon learn that Plato is subtly instructing us as to what we need to expect. It is, in particular, relevant the fact that there is no fourth guest.

The dialogue is rich in imagery, and in mathematical ideas. For example, this is one of the two dialogues (the second being *Critias*) in which we read of the myth of Atlantis, that magical kingdom whose existence was never proved and which, Plato tells us, disappeared somewhere in the ocean. It is also the dialogue in which Plato describes and uses what we now refer to as the five Platonic solids (which Euclid studies in the XIII book of his *Elements*).

But the portion of the dialogue with which we are mostly concerned is the mythological description that Plato uses to describe how the universe was created. Before we begin the retelling of the story, we should remind the reader that whenever Plato was using a myth, he intended it to be an *εἰκὼς μῦθος* namely a likely tale. In other words, he knew that the story itself was not true, in the traditional sense of the world, but he intended it to be metaphorically true. It is within this intellectual framework that we need to explore all of Plato's myths, and certainly the one in which the creation of the universe is described.

So, when Plato describes the creation of the universe, he begins with the demiurge, who picks up a material that he himself assembles (the discussion of how he does this would take us to far astray), and cuts it up as we read in the passage below:

[4] Boethius, *De institutione musica libri quinque*, G. Friedlein (ed.), B.G. Teubner, Lipsiae, 1867, pp. 175-371: pp.197-198. See H. Chadwick, *Boethius. The Consolations of Music, Logic, Theology, and Philosophy*, Clarendon Press, Oxford, 1981, pp. 84-90.
[5] Plato, *Timaeus*, 17a. Trans. by R.G. Bury, Harvard University Press, *et al.*, Cambridge (MA), 1961, p. 17.

> And He began making the division thus: First He took one portion from the whole; then He took a portion double of this; then a third portion, half as much again as the second portion, that is, three times as much as the first; the fourth portion He took was twice as much as the second; the fifth three times as much as the third; the sixth eight times as much as the first; and the seventh twenty-seven times as much the first.
>
> After that He went on to fill up the intervals in the series of the powers of two and the intervals in the series of powers of three in the following manner:
>
> He cut off yet further portions from the original mixture, and set them in between the portions above rehearsed, so as to place two Means in each interval, - one a Mean which exceeded its Extremes and was by them exceeded by the same proportional part or fraction of each of the Extremes respectively; the other Mean which exceeded one Extreme by the same number or integer as it was exceeded by its other Extreme. And whereas the insertion of these links formed fresh, that is to say, intervals of 3:2 and 4:3 and 9:8, He went on to fill up the 4:3 intervals with 9:8 intervals. This still left over in each case a fraction, which is represented by the terms of the numerical ratio 256:243.
>
> And thus the mixture, from which He had been cutting these portions off, was now all spent.[6]

Let us clarify here for the reader a few of the steps that the demiurge takes. First we see how he builds the powers of two and the powers of three. Note that the only powers used are the zero-th power (what he calls *the whole* since the zero-th power of any number is always one), and then the first, second, and third power. This is consistent with the opening of the dialogue, where only three guests appear, and with the way in which the Greeks thought about measures and powers. Specifically, the zero-th power represents the point, while the first power is a measure of length (two or three in this case). Then the second power represents the square with assigned side (four or nine in Plato's example) and finally the third power represents the cube with assigned side (eight and twenty seven here). Unlike for us moderns, for whom the process of taking higher powers is an abstract process that allows us to take arbitrarily high powers, for the Greeks the idea of taking powers higher than three made no sense, because numbers only made sense in so much as they represented geometrical realities. And, for the Greeks (and to some extent for us as well) there is no geometric reality in dimension higher than three.

A second clarification that needs to be made, deals with the two kinds of means that Plato introduces. One, he says, is "exceeding and exceeded by equal parts of its extremes." To understand what he means, let a and b be the two extremes. Then the mean c which satisfies the request above needs to be such that

$$c = a + xa = b - xb$$

where x is the equal part which needs to be determined. By solving $a+xa = b-xb$ we find that $x = (b - a)/(a + b)$, and this gives us $c = 2ab/(a + b)$.

If, for example, we set $a = 1$ and $b = 2$, we obtain, as indicated in the dialogue, that this mean (usually called the harmonic mean) is $c = 4/3$.

The second mean introduced by Plato is the so-called arithmetic mean, and it is the mean that "exceeds and is exceeded by an equal number". This means that if a and b are the two extremes, then the arithmetic mean c is such that

$$c = a+x = b-x$$

[6] Plato, *Timaeus*, 35b-36c. *Op.cit.*, pp. 67-71.

which easily gives $c = (a + b)/2$; in the case of $a = 1$ and $b = 2$, we immediately obtain $c = 3/2$.

This shows us how the "filling" process that Plato indicates takes place. When we consider the powers of two, the means between 1 and 2 are, respectively 4/3 and 3/2, between 2 and 4 they are 8/3 and 3, and finally between 4 and 8 they are 16/3 and 6. Similarly, when we consider the powers of 3, we have the means 3/2 and 2 between 1 and 3, the means 9/2 and 6 between 3 and 9, and finally the means 27/2 and 18 between 9 and 27.

We now need to interpret what Plato says when he talks about filling up "all the intervals of 4/3 with the interval of 9/8, leaving a fraction over; and the interval which this fraction expressed was in the ratio of 256 to 243." So, here it is what Plato suggests to fill the intervals up:

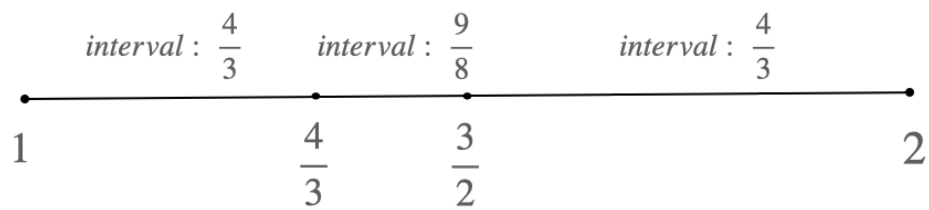

Figure 3.
Harmonic and arithmetic mean: the two 4/3 intervals and the 9/8 interval.

- consider the interval [1,2], and insert in their places the harmonic mean 4/3 and the arithmetic mean 3/2 of the extremes of the interval, obtaining the sequence

$$1 \qquad 4/3 \qquad 3/2 \qquad 2$$

- notice that both [1,4/3] and [3/2,2] are intervals of 4/3;
- notice that the middle interval [4/3,3/2] is an interval of 9/8;
- "fill up the interval [1,4/3] of 4/3 with the interval of 9/8, leaving a fraction over", i.e., take the first two powers of 9/8 and approximate the third power by taking 4/3.

$$1 \qquad 9/8 \qquad (9/8)^2 = 81/64 \qquad 4/3$$

As one can easily see, the ratio (4/3)/(81/64) gives

$$(4 \times 64)/(3 \times 81) = 256/243$$

the famous magic number Plato exhibits.[7]

- continue and fill the interval [3/2,2], with a number x such that $x/(3/2) = 9/8$, i.e $x = 27/16$, and a number y satisfying $y/(27/16) = 9/8$, that is $y = 243/128$. Now we have $2/(243/128) = 256/243$ and the magic number appears once more. We have now obtained the sequence

[7] Let us note that the numbers 256/243 and 9/8 are quite close since 256/243 ~ 1.05349794239 and 9/8 = 1.125.

3/2   27/16   243/128   2

In conclusion the sequence that Plato identifies, deliberately considering only the numbers (and the powers of) 1,2 and 3, is:

(2.1)

1   9/8   81/64   4/3   3/2   27/16   243/128   2

Corresponding to

(2.2)

$$1 \quad 9/8 \quad (9/8)^2 \quad (256/243) \times (9/8)^2 \quad (256/243) \times (9/8)^3$$
$$(256/243) \times (9/8)^4 \quad (256/243) \times (9/8)^5 \quad (256/243)^2 \times (9/8)^5$$

This structure is really surprising!

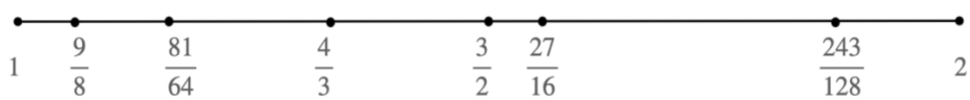

Figure 4.

To have a better feeling of what is happening, we now pass, in modern terms, to the logarithms of the involved terms, transforming a "geometric sequence" into a representative "arithmetic sequence". Using logarithms in basis 2 we obtain the following picture:

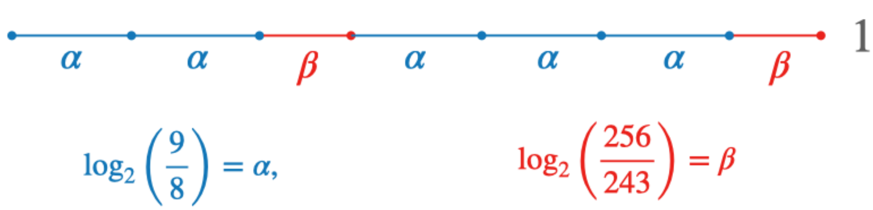

Figure 5.

Most readers will be baffled by the process outlined by Plato. What has all this numerical stuff have to do with the creation of the universe? Where are these very specific (and fairly large) fractions coming from? If this were written by anybody else, we would certainly feel authorized to dismiss the entire thing as some mysterious numerology. But this is Plato writing, and for sure he must have a specific idea in mind. This is where the music comes into the picture. Indeed, while these numbers may make little sense to the general reader, a student of harmony (even a beginner) will recognize that the ratios that appear are nothing but the ratios necessary to build the twelve tone scale that we customarily use. Indeed, if one looks carefully at the sequence of fractions, one will see an interesting pattern: there are three numbers separated by the same fraction, and then four numbers.

So, it is clear that what we have in front of us is a mathematical representation of what the Greeks called the monochord, and whose appearance is just like the keyboard of a piano (with two black keys, followed by three black keys, in a repeating pattern).

Figure 6.

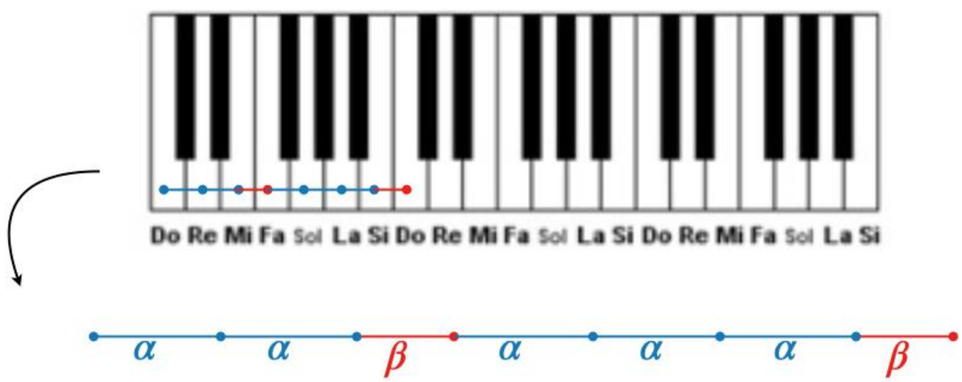

Figure 7.

The next surprise is that we can continue with the subsequent interval of 2, i.e., [2,4]. The harmonic mean is now 8⁄3 and the arithmetic mean is 3. In the sequence

$$2 \qquad 8/3 \qquad 3 \qquad 4$$

again [8⁄3,3] is an interval of 9⁄8, and [2,8⁄3] together with [3,4] are both intervals of 4⁄3. In this manner, with the same approach as before, Plato obtains the sequence:

$$2 \quad 9/4 \quad 81/32 \quad 8/3 \quad 3 \quad 27/8 \quad 243/64 \quad 4$$

which corresponds to

$$\begin{array}{lll} [(256/243)]^2 \times (9/8)^5 & [(256/243)]^2 \times (9/8)^6 & \\ [(256/243)]^2 \times (9/8)^7 & [(256/243)]^3 \times (9/8)^7 & [(256/243)]^3 \times (9/8)^8 \\ [(256/243)]^3 \times (9/8)^9 & [(256/243)]^3 \times (9/8)^{10} & [(256/243)]^4 \times (9/8)^{10} \end{array}$$

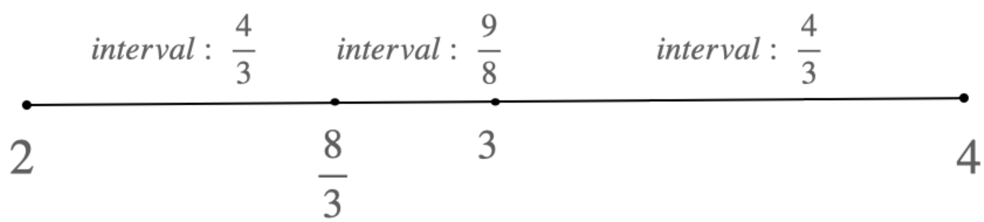

Figure 8.
Harmonic and arithmetic mean: the two 4⁄3 intervals and the 9⁄8 interval in the interval [2,4].

Indeed, for any natural number n, we can treat in the same manner any interval of 2, i.e., $[2^n, 2^{n+1}]$. The harmonic mean is now $2^n(4/3)$ and the arithmetic mean is $2^n(3/2)$. In the sequence

$$2^n \quad 2^n(4/3) \quad 2^n(3/2) \quad 2^{n+1}$$

again $[2^n(4/3),2^n(3/2)]$ is an interval of 9⁄8, and $[2^n,2^n(4/3)]$ together with $[2^n(3/2),2^{n+1}]$ are both intervals of 4⁄3. In this manner, with the same approach as before, the Plato method leads to the sequence:

(2.3)

$$2^n \quad 2^n(9/8) \quad 2^n(81/64) \quad 2^n(4/3) \quad 2^n(3/2) \quad 2^n(27/16) \quad 2^n(243/128) \quad 2^{n+1}$$

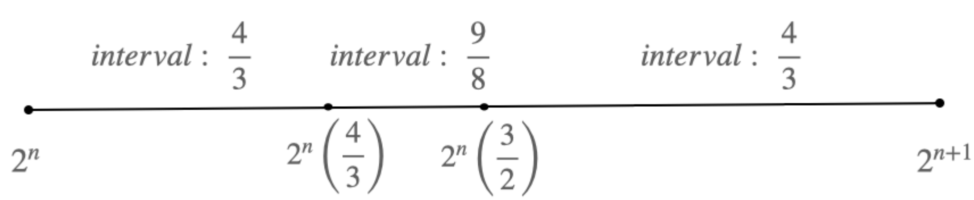

Figure 9.
Harmonic and arithmetic mean: the two 4⁄3 intervals and the 9⁄8 interval in $[2^n,2^{n+1}]$.

At this point we can extend the pattern of the piano (see Figure 7) to the entire keyboard.

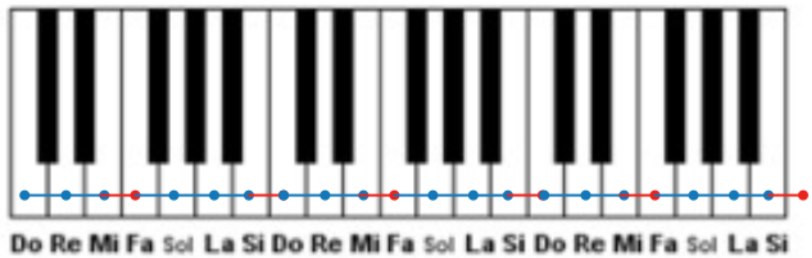

Figure 10.

If we now turn our attention to the case of the sequence of powers of 3, i.e., to begin with, to the case of the interval [1,3], and we insert the harmonic and arithmetic mean, we obtain the sequence

1 3⁄2 2 3

which belongs to the sequence found while filling the intervals of 2 with intervals of 9⁄8. Notice that this sequence has two intervals of 3⁄2 and one inteval of 4⁄3.

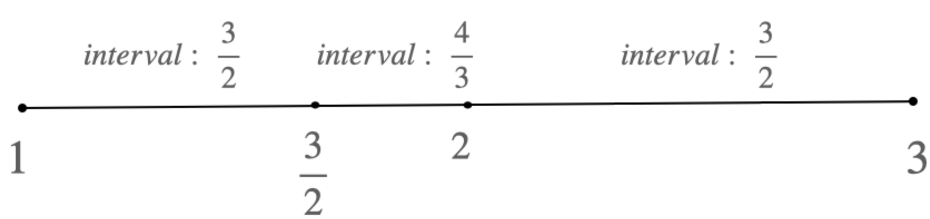

Figure 11.
Harmonic and arithmetic mean: the two 3⁄2 intervals and the 4⁄3 interval in [1,3].

Plato already suggested how to fill the intervals of 4⁄3 and 3⁄2 with intervals of 9⁄8 and 256⁄243!! As a consequence we can fill the interval [1,3] as follows:

1 9⁄8 81⁄64 4⁄3 3⁄2 27⁄16 243⁄128 2 9⁄4 81⁄32 8⁄3 3

As in the case of the powers of 2, one can treat in the same manner any interval of 3, i.e., $[3^n, 3^{n+1}]$. The harmonic mean is now $3^n(3/2)$ and the arithmetic mean is
$3^n 2$. In the sequence

3^n $3^n(3/2)$ $3^n 2$ 3^{n+1}

again $[3^n(3/2),3^n2]$ is an interval of 4/3, and $[3^n,3^n(3/2)]$ together with $[3^n2,3^{n+1}]$ are both intervals of 3/2. Now, using the filling (2.3) for different n, we obtain, up to $27 = 3^3$, the sequence

1	9/8	81/64	4/3	3/2	27/16	243/128	2
2	9/4	81/32	8/3	3	27/8	243/64	4
4	9/2	81/16	16/3	6	27/4	243/32	8
8	9	81/8	32/3	12	27/2	243/16	16
16	18	81/4	64/3	24	27	[243/8	32]

where we coulored the intervals of 3 and their harmonic and arithmetic means. This sequence appears in the *Timaeus*, and it is intended to correspond to the notes of a musical scale, up to four octaves.[8]

This process shows that the Greeks understood that if we take a tout piece of string and pluck it, we will obtain a specific sound, whose pitch depends from the length of the string, and that if we hold the string in the middle, and we pluck only one of its halves, we will get exactly the same pitch, but (as we say in modern terms) at a higher octave (same pitch, but more acute).[9]

Thus, if we want to build a scale of notes between two equal pitches (of different octaves), we need to find a constant ratio that will allow us to go from a string to its double. Effectively, we need to find a fraction x, and an integer n, such that x to the power n will be 2. In this case we will have a scale with n tones or semitones, and the ratio between each tone will be x.

This is where the Greeks (and the Pythagoreans first) identified a central problem, one that apparently bothered them significantly. Indeed, to find a fraction whose square is 2 is impossible.[10] Even worse, the same argument shows that it is impossible to find a fraction whose n-th power (regardless of what n is) is 2, that is, all the roots of 2 are irrational numbers.

The musical consequence of this mathematical theorem is the fact that it is impossible to split the string in portions of constant proportions, and fill it all (in other words a perfect temperament is impossible). If we try, we will always come up with some little extra piece that is necessary to complete the process. So, the Greeks set out to figure out what would be the integer n such that a split into n steps would most closely represent the entire string.

As we have seen, Plato in the *Timaeus* proposed to split the string in two intervals of 9/8, followed by one interval of 256/243, followed by three intervals of 9/8 followed by an interval of 256/243, for a total of seven intervals of two different ratios. In practice the number 2 was realised as

(2.4)
$$2 = 9/8 \times 9/8 \times 256/243 \times 9/8 \times 9/8 \times 9/8 \times 256/243$$

and this splitting corresponds to the seven notes of the natural scale:

[8] Plato, *Timaeus*, 36b-c. *Op.cit.*, pp. 70-71.
[9] The specific determination of the behavior of a vibrating string was one of the fundamental objects of study in the early part of modern mathematics, and became the inspiration for the study of several differential equations and ultimately Fourier Analysis.
[10] Suppose, by contradiction, that two integers p and q have no common integer divisors and are such that $(p/q)^2 = 2$. By rewriting the previous equality as $p^2 = 2q^2$ we deduce that the number 2 divides the integer p^2, and hence that it divides p. Therefore $p = 2r$ for some integer r, and hence $(2r)^2 = 2q^2$, i.e., $4r^2 = 2q^2$ and $2r^2 = q^2$. As before we deduce that 2 divides q, and hence that p and q have 2 as a common divisor. Contradiction.

DO, RE, MI, FA, SOL, LA, SI

When numbers move, they always produce surprises! The interval 9⁄8 turns out to be approximately twice the proportion (i.e, to be approximately the square) of the interval of 256⁄243. In fact the ratio

$$(9/8)/(256/243)^2 = 1.0136432647705078$$

called *Pythagorean comma*, is very close to 1 and states the two latter intervals are really very close to each other. This is the reason why, in music, the interval 9⁄8 is called *tone* and the interval 256⁄243 is called *semitone* (even if, as we already pointed out, this is so only approximately). In the next few pages, for the sake of clarity, we will denote 256⁄243 (also called the *Pythagorean leimma*) by the Greek letter λ, and the ratio $(9/8)/\lambda = (2187/2048)$ by μ, so that $\lambda \times \mu = 9/8$. The quantity μ is commonly called *chromatic semitone*.

If we now, in equation (2.4), count 2 semitones for any tone, we end up counting a total of 12 semitones to cover the interval [1,2]. With the language already used, it turns out that $n = 12$ is a good choice as the number of interval in which a string should be splitted into; this kind of splitting indeed corresponds to the twelve notes of the dodecaphonic scale:

DO, DO♯, RE, RE♯, MI, FA, FA♯, SOL, SOL♯, LA, LA♯, SI

The twelve notes, easily identified on the keyboard of any piano, may correspond to the splitting:

$$1 \times \lambda \times \mu \times \lambda \times \mu \times \lambda \times \lambda \times \mu \times \lambda \times \mu \times \lambda \times \mu \times \lambda$$

On the other hand, the twelve notes

DO, RE ♭, RE, MI ♭, MI, FA, SOL ♭, SOL, LA ♭, LA, SI ♭, SI

may correspond to the splitting [11]

$$1 \times \mu \times \lambda \times \mu \times \lambda \times \lambda \times \mu \times \lambda \times \mu \times \lambda \times \mu \times \lambda \times \lambda$$

But of course different approximations can be (and actually were) used to split the string [1,2] in 12 “proportional” intervals. One could also use for instance the splitting suggested by the fact that $(9/8)^6 > 2$ is the first power of 9⁄8 that exceeds the number 2, and that (as we have seen)

$$2 = (9/8)^5 \times (256/243)^2$$

[11] Note that we use the sharp symbol ♯ to indicate that a note is raised of the semitone λ, and the flat symbol ♭ to denote that a note is lowered of the same semitone. However, since $\lambda\times\mu = 9/8$ but $\lambda \neq \mu$, the notes DO♯ and RE ♭, for example, are slightly different. This is the case, at least, in any non-equal temperament, while if we were to consider equal temperament (such as we do on a piano) the two notes would end up being the same, and the different notation is only used as a matter of convention.

The corresponding splitting could be

$$1 \times \lambda \times \mu \times \lambda \times \mu \times \lambda \times \mu \times \lambda \times \mu \times \lambda \times \mu \times \lambda \times \lambda$$

where the two-step symmetry is only broken at the very last step.

Among the many possible mathematical tunings, the following one is known as the *Pythagorean tuning* of the twelve notes of the Western chromatic scale:

$$1 \times \lambda \times \mu \times \lambda \times \mu \times \lambda \times \mu \times \lambda \times \lambda \times \mu \times \lambda \times \mu \times \lambda$$

This entire process, thus, shows that what is called the "perfect temperament" of the scale is impossible, that a dodecaphonic scale is pretty accurate, and that the difference between the scale and the perfect temperament can be measured precisely.

Of course we should point out that a perfect temperament of the [1,2] string can theoretically be implemented with the use of irrational numbers: the interval $x = \sqrt[12]{2}$ is (by definition) such that $x^{12} = 2$, and gives a perfect splitting in 12 equally proportional parts of the string [1,2].

3. Zarlino temperament

In this paper we have not attempted to trace the history of temperament, but rather to discuss what is probably the first written record of an attempt to create a good temperament.

The history of temperament is, of course, much richer, and certainly not the goal of this paper. Indeed, following what we have written in the previous section, it remains clear that the only way to have a perfect separation of the octave in twelve equally distant notes, requires the calculation of the twelfth root of two. Since such number is irrational, the Pythagoreans created the scale based on the use of the numbers two and three, and their powers. But this scale becomes problematic when the musician moves over several octaves along cycles of thirds and fifths. This forced musicians, over the next several centuries, to come up with many different variations. Of significant note is the work of Gioseffo Zarlino, who follows the spirit of the Pythagoreans, but allows the use of the number five in the fractions that describe the scale.[12] As a result he replaces some of the high powers in the scales (2.1) and (2.2) with lower powers including five.

[12] Gioseffo Zarlino, Italian music theorist and composer, his most important works are *Le Istitutioni Harmoniche*, Venezia, 1558 and *Le Dimostrationi Harmoniche*, Francesco dei Franceschi Senese, Venezia, 1571. See G. Mambella, *Corpo sonoro, geometria e temperamenti. Zarlino e la crisi del fondamento numerico della musica*, in *Music and Mathematics*, P. Vendrix (ed.), Brepols Publishers, Turnhout, 2008, pp. 185-234.

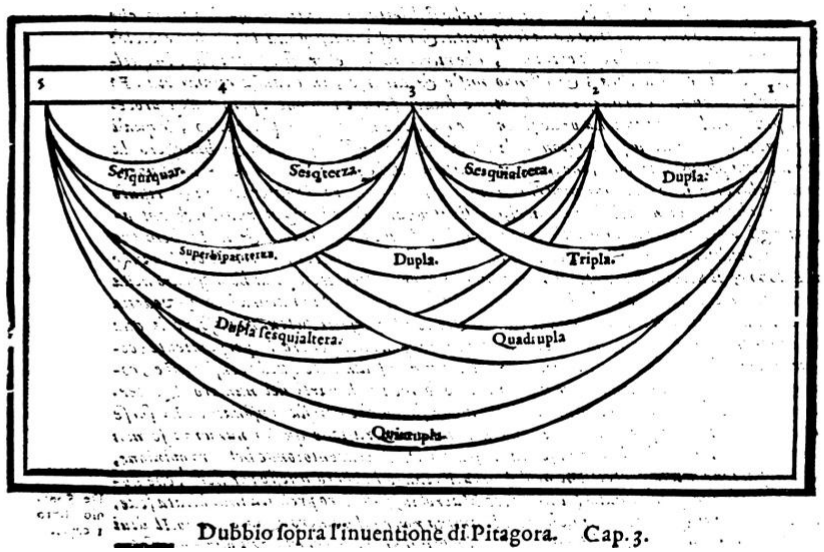

Figure 12.
Zarlino admits number five among the Pythagorean numbers one, two, three, and titles this image and the chapter *Doubt on the invention of Pythagoras* (G. Zarlino, *Le Istitutioni Harmoniche*, Venezia, 1558, p. 72.)

Specifically, he replaces 81⁄64 (or $3^4/2^6$) with 5⁄4 (the ratio between these two numbers - the *Zarlino comma* - being 81⁄80, so that the approximation is actually quite good).[13] Since Zarlino wants to keep the harmonic and arithmetic means as suggested in Plato's work, this approximation forces the use of three intervals: the *major tone* $T = 9/8$, the *minor tone* $T' = 10/9$ and the *diatonic semitone* $t = 16/15$. This is a particularly Pythagorean choice because the Pythagoreans preferred fractions in which the numerator exceeds the denominator of one. Once these intervals are accepted, the Zarlino temperament becomes:

$$1 \quad 9/8 \quad 5/4 \quad 4/3 \quad 3/2 \quad 5/3 \quad 15/8 \quad 2$$

$$1 \times T \times T' \times t \times T \times T' \times T \times t$$

Still in the Pythagorean spirit, the *just temperament* played a role in the search for the best possible splittings of the interval of the string.[14] This tuning is not different in spirit and intervals from the one of Zarlino, and has the same three intervals: the *major tone* $T = 9/8$, the *minor tone* $T' = 10/9$ and the *diatonic semitone* $t = 16/15$. The just temperament uses the fraction 6⁄5 in place of 5⁄4, and results in

$$1 \quad 9/8 \quad 6/5 \quad 4/3 \quad 3/2 \quad 5/3 \quad 16/9 \quad 2$$

[13] See C.V. Palisca, *Music and Ideas in the Sixteenth and Seventeenth Centuries*, University of Illinois Press, Urbana and Chicago, 2006, partic. chap. III.
[14] *Music and Mathematics. From Pythagoras to Fractals*, J. Fauvel, R. Flood and R. Wilson (eds.), Oxford University Press, Oxford, 2003, p. 24.

$$1 \times T \times t \times T' \times T \times T' \times t \times T$$

The name of *fundamental just (dodecaphonic) scale* indicates a temperament of the twelve notes of the first octave, which seems to use both the just scale and the Zarlino scale. The quantity $s = (25/24)$ is the *chromatic semitone* such that $t \times s = T'$, i.e, such that its product with the semitone t equals the minor tone T'. Further chromatic semitones are the quantities $r = (27/25)$ and $p = (135/128)$, such that respectively, $s \times r = T$ and $t \times p = T$. Here is the description of this scale:[15]

1 (25/24) 9/8 6/5 5/4 4/3 (45/32) 3/2 8/5 5/3 9/5 (15/8) 2

$$1 \times s \times r \times t \times s \times t \times p \times t \times t \times s \times r \times s \times t$$

which corresponds to

$$1 \times (s \times r) \times t \times (s \times t) \times (p \times t) \times (t \times s) \times (r \times s) \times t$$

$$1 \times T \times t \times T' \times T \times T' \times T \times t$$

It might be significant to notice that one can actually move the numbers above to obtain a different scale, let us say a *pure just dodecaphonic temperament*. With a change in the two last digits of the previous splitting, we obtain:

1 (25/24) 9/8 6/5 5/4 4/3 (45/32) 3/2 8/5 5/3 (16/9) (48/25) 2

$$1 \times s \times r \times t \times s \times t \times p \times t \times t \times s \times t \times r \times s$$

which corresponds to

$$1 \times (s \times r) \times t \times (s \times t) \times (p \times t) \times (t \times s) \times t \times (r \times s)$$

$$1 \times T \times t \times T' \times T \times T' \times t \times T$$

i.e., to the just scale.

4. A glance ahead. Conclusions

Among Zarlino's students, one finds the lutenist Vincenzo Galilei (more famous because of his son Galileo), who used 18/17 as an approximation for the twelfth root of two.[16] Zarlino and Galilei had a

[15] See O.H. Jorgensen, *Tuning: containing the perfection of Eighteenth-century temperament, the lost art of Nineteenth-century temperament, and the science of equal temperament. Complete with instructions for aural and electronic tuning*, Michigan State University Press, East Lansing (MI), 1991. See also <https://pages.mtu.edu/ suits/Physicsofmusic.html>.

[16] Vincenzo Galilei, Italian lutenist, composer, and music theorist, author of *Dialogo di Vincentio Galilei della musica antica et della moderna*, G. Marescotti, Fiorenza, 1581 and of *Discorso di Vincentio Galilei intorno alle opere di Gioseffo Zarlino et altri importanti particolari attenenti alla musica*, G. Marescotti, Fiorenza, 1589.

series of very strong disagreements on what the appropriate scale should be (partially because there is no correct answer to the problem) as it is extensively discussed in Walker.[17]

Their work, however seems to have influenced the next step, namely the temperament suggested by Simon Stevin, who also calculated an approximation of the root of two, in the attempt to create an equal temperament.[18]

The race to better approximations for the roots of two, however, ended up being somewhat idle, since an equal temperament would not be musically interesting, as it would make all keys equal to each other, thus removing what musicians refer to as key-colour or key-character. It is for this reason that musicologists believe that Bach, in his *Well Tempered Klavier* (appropriately called Well Tempered instead of Equal Tempered) did not use an equal temperament (which in any case would have been impossible to obtain with the technological means available to him) but rather a temperament that allowed him to take full advantage of the different key colours.[19]

We conclude by remarking on the initial comments in this paper: music develops and appears as we permit numbers (fractions in this case) to acquire a dynamical aspect and create, through their growth, the various keys that allow the richness of the musical texture. This idea was adumbrated in Plato’s work, but its importance to his philosophical worldview cannot be underestimated, as it is a presence in most of his dialogues (though we focused on *Timaeus*, we could have made a similar analysis for, say, *The Republic*), and his intuition remains as a guide for those interested in understanding the evolution of musical aesthetics through the centuries.

That music was the outcome of an interweaving of knowledge was an established awareness already in the early modern age, and that the musician should be the repository of multiple knowledge and experience is brilliantly testified to in his Compendium of Music by the philosopher and mathematician René Descartes:

> To a *Complete Musitian* [...] is required a more the superficial insight into all kinds of Humane Learning. For, He must be a *Physiologist*; that He may demonstrate the Creation, Nature, Proprieties, and Effects of a Natural Sound. A *Philologer*, to inquire into the first Invention, Institution, and succeding Propagation of an Artificial Sound, or Musick. An *Arithmetician*, to be able to explaine the Causes of Motions Harmonical, by Numbers, and declare the Mysteries of the new Algebraical Musick. A *Geometrician*, to evince in great variety, the Original of Intervalls Consono-dissonant, by the Geometrical, Algebraical, Mechanichal Division of a Monochord. A *Poet*, to conform his Thoughts, and Words to the Lawes of precise Numbers, and distinguish the Euphonie of Vowells and Syllables. A *Mechanique*, to know the exquisite Structure of Fabrick of all Musical Instruments, Winde, Stringed, or Tympanous *alias* Pulsatile. A *Metallist*, to explore the different Contemperations of Barytonous and Oxytonous. Or Grave and Acute toned Metalls, in order to the Casting of tuneable Bells, for Chimes, etc. An *Anatomist*, to satisfie concerning the Manner, and Organs of the Sense of Hearing. A *Melothetick*, to lay down a demonstrative method for the Composing, or Setting of all Tunes, and

[17] D.P. Walker, “Some Aspects of the Musical Theory of Vincenzo Galilei and Galileo Galilei”, *Proceedings of the Royal Musical Association*, 100, 1 (1973): pp. 33-47.

[18] R. Rasch, “Simon Stevin and the Calculation of Equal Temperament”, in *Music and Mathematics*, cit., pp. 253-320; H. Floris Cohen, “Simon Stevin’s equal division of the octave”, *Annals of Science* 44, 5 (1987): pp. 471-488.

[19] J.S. Bach, *The Well-Tempered Clavier*, BWV 846-893, (*Das wohltemperierte Klavier*), collection of 48 preludes and fugues, published in two books (1722 and 1742). *Prelude n.1*, C major BWV 846, was played by Bach in Pythagorean tuning. See also A. Frova, “Quale temperamento per Bach?”, *Rivista Italiana di Musicologia* 41, 1 (2006): pp. 167-180.

Ayres. And, lastly, He must be so far a *Magician*, as to excite Wonder, with reducing into Practice the Thaumaturgical, or admirable Secrets of Musick.[20]

[20] R. Descartes, *Compendium of Musick*, trans. William Lord Brouncker, Thomas Harper, London, 1653, *The stationer to the ingenious reader*.